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AUG 76 G E ANDREWS, R ASKEY DAAG29-75-C-0024
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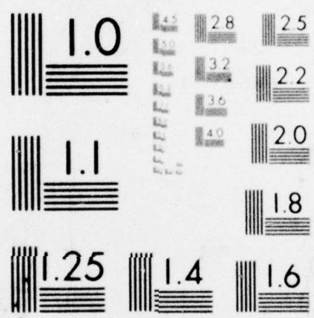

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See tag

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A SIMPLE PROOF OF RAMANUJAN'S SUMMATION OF THE ${}_1\psi_1$.

George E. Andrews and Richard Askey

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ABSTRACT

A simple proof by functional equations is given for Ramanujan's ${}_1\psi_1$ sum. Ramanujan's sum is a useful extension of Jacobi's triple product formula, and has recently become important in the treatment of certain orthogonal polynomials defined by basic hypergeometric series.

AMS(MOS) Subject Classification: 33A25, 33A30

Key Words: Basic hypergeometric functions, Theta functions, Ramanujan sum, Jacobi triple product.

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A SIMPLE PROOF OF RAMANUJAN'S SUMMATION OF THE ${}_1\psi_1$

George E. Andrews(1) and Richard Askey(2)

In [5; p. 222, eq. (12.12.2)] G. H. Hardy alludes to Ramanujan's "... remarkable formula with many parameters.":

$$(1) \qquad \sum_{n=-\infty}^{\infty} \frac{(a;q)_n x^n}{(b;q)_n} \equiv {}_1\psi_1\binom{a;q,x}{b}$$

$$= \frac{(b/a,q)_\infty (q;q)_\infty (q/ax;q)_\infty (ax;q)_\infty}{(b;q)_\infty (b/ax;q)_\infty (q/a;q)_\infty (x;q)_\infty},$$

where $\left|\frac{b}{a}\right| < |x| < 1$, $|q| < 1$, $(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n)$, and $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$.

There are four published proofs of this result ([1], [2], [4] and [7]). Those in [1], [2] and [7] rely on somewhat tricky rearrangement of series and on the q-analog of Gauss's summation [10; p. 97, eq. (3.3.2.5)]

$$(2) \qquad \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n \left(\frac{c}{ab}\right)^n}{(c;q)_n (q;a)_n} = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty},$$

where $|c| < \min(1, |ab|)$. The other proof uses the q-analogue of the binomial series [10; p. 92, eq. (3.2.2.11)]:

$$(3) \qquad \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}, \quad |t| < 1, \quad |q| < 1,$$

but it is far from simple. Since Ramanujan's summation (1) has recently become important in the treatment of certain orthogonal polynomials defined

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by basic hypergeometric series [3], it has become worthwhile to present an almost trivial proof of (1). Another very simple proof has been found by M. Ismail [6].

Proof of (1). We begin by noting that for $|q| < 1$, $f(b) \equiv {}_1\psi_1\left({a;q,x \atop b}\right)$ is an analytic function of b inside $|b| < \min(1, |ax|)$, since

$$(4) \qquad f(b) = \sum_{n=0}^{\infty} \frac{(a;q)_n x^n}{(b;q)_n} + \sum_{n=1}^{\infty} \frac{(1-\frac{b}{q^n})\ldots(1-\frac{b}{q})x^{-n}}{(1-\frac{a}{q^n})\ldots(1-\frac{a}{q})} .$$

Furthermore,

$$(5) \qquad {}_1\psi_1\left({a;\,q,x \atop b}\right) - a\,{}_1\psi_1\left({a;\,q,qx \atop b}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a;q)_{n+1}x^n}{(b;q)_n} = x^{-1}\left(1-\frac{b}{q}\right)\sum_{n=-\infty}^{\infty}\frac{(a;q)_{n+1}x^{n+1}}{(\frac{b}{q};q)_{n+1}}$$

$$= x^{-1}\left(1-\frac{b}{q}\right){}_1\psi_1\left({a;q,x \atop b/q}\right) .$$

Hence

$$(6) \qquad f(bq) - x^{-1}(1-b)f(b) = a\sum_{n=-\infty}^{\infty}\frac{(a;q)_n q^n x^n}{(bq;q)_n}$$

$$= -a\,b^{-1}\sum_{n=-\infty}^{\infty}\frac{(a;q)_n(1-bq^n-1)x^n}{(bq;q)_n} = -ab^{-1}(1-b)\,f(b) + ab^{-1}\,f(bq) ,$$

and so

$$\left(1-\frac{a}{b}\right)f(bq) = (1-b)(x^{-1}-ab^{-1})\,f(b) ,$$

or

$$(7) \qquad f(b) = \frac{(1-\frac{b}{a})}{(1-b)(1-\frac{b}{ax})}\,f(bq) .$$

If we iterate (7) $n-1$ times we find that

$$(8)\qquad f(b) = \frac{(b/a\,;q)_n}{(b;q)_n (b/ax;q)_n} f(bq^n) ,$$

and since $f(b)$ is analytic in the neighborhood of 0 given by $|b| < |ax|$, we obtain in the limit as $n \to \infty$,

$$(9)\qquad f(b) = \frac{(b/a;\, q)_\infty \; f(0)}{(b;q)_\infty (b/ax;q)_\infty} .$$

Now we observe from (4) and (3) that

$$(10)\qquad f(q) = \sum_{n=0}^{\infty} \frac{(a;q)_n x^n}{(q;q)_n} = \frac{(ax,\, q)_\infty}{(x;q)_\infty} .$$

This allows us to evaluate $f(0)$ by setting $b = q$ in (9) :

$$(11)\qquad f(0) = \frac{(q;q)_\infty \left(\frac{q}{ax};\, q\right)_\infty \; f(g)}{(q/a;\, q)_\infty}$$

$$= \frac{(q;q)_\infty \left(\frac{q}{ax};q\right)_\infty (ax;q)_\infty}{(q/a;q)_\infty \; (x;q)_\infty} .$$

Finally we may utilize (11) to eliminate $f(0)$ from (9) :

$$(12)\qquad {}_1\psi_1\left(\begin{matrix} a ; q, x \\ b \end{matrix}\right) = f(b) = \frac{(b/a;q)_\infty \; (q;q)_\infty \; (q/ax;q)_\infty \; (ax;q)_\infty}{(b;q)_\infty \; (b/ax;q)_\infty \; (q/a,q)_\infty \; (x;q)_\infty} ,$$

as desired.

Note that Jacobi's triple product identity follows directly from (1) if we replace a by α^{-1}, x by $z\alpha$ and then set $\alpha = b = 0$:

$$(13)\qquad \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n = (q;q)_\infty \; (q/z;q)_\infty \; (z;q)_\infty .$$

I. J. Schoenberg has pointed out an interesting property of $\frac{(a;q)_n}{(b;q)_n}$ which follows from Ramanujan's sum. A sequence a_n, $n = 0, \pm 1, \ldots$, is said to be *totally positive* if all subdeterminants of the doublely infinite matrix $A = (a_{i-j})_{-\infty<i,j<\infty}$ are nonnegative. Schoenberg [9] proved that a sequence a_n is totally positive if the bilateral generating function $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ has the representation

$$f(z) = e^{cz+dz^{-1}} \prod_{i=1}^{\infty} \frac{(1+\alpha_i z)(1+\delta_i z^{-1})}{(1-\beta_i z)(1-\gamma_i z^{-1})}, \tag{14}$$

$$c, d, \alpha_i, \beta_i, \gamma_i, \delta_i \geq 0, \quad \sum_{i=1}^{\infty} (\alpha_i + \beta_i + \gamma_i + \delta_i) < \infty,$$

in the interior of an annulus centered at the origin.

If $a < b < 0$ in (1) then the generating function has the form (14) and so

$$a_n = \frac{(a;q)_n}{(b;q)_n} = \prod_{k=0}^{\infty} \frac{(1-bq^{k+n})(1-aq^k)}{(1-aq^{k+n})(1-bq^k)}$$

is a totally positive sequence for $a < b \leq 0$, $0 < q < 1$. Schoenberg [9] proved this when $b = 0$. For an extended discussion of totally positive sequences see Karlin [8].

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